PERIOD RELATIONS FOR HYPERELLIPTIC RIEMANN SURFACES

BY

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ABSTRACT

In this paper, we derive an elementary derivation of the Schottky relation for hyperelliptic Riemann surfaces of genus 4. The relation obtained generalizes immediately to hyperelliptic surfaces of genus greater than 4. The derivation is elementary in the sense that it does not require the Schottky-Jung conjecture.

Introduction

If S is a compact Riemann surface of genus g, $g \geqq 4$, and if Γ,Δ $\Gamma = \gamma_1,\cdots,\gamma_g$ $\Delta = \delta_1,\cdots,\delta_g$ is a canonical homology basic on S, then one can choose a basis for the abelian differentials of first kind $\phi_1,\cdots,\phi_g$ dual to Γ,Δ i.e.: $\int_{\gamma_j}\phi_i = \delta_{ij}$ and form the period matrix (I,Π) where $\Pi = (\pi_{ij})$ and $\pi_{ij} = \int_{\delta_j}\phi_i$. It is well known that the matrix Π formed in this fashion is an element of $\mathfrak{S}_g$, the Siegel upper half plane of degree g, i.e.: Π is symmetric and the imaginary part of Π is positive definite.

Recently Rauch [9, 10] proved that the $(g(g+1))/2$ elements of Π are holomorphic functions of $3g-3$ complex variables and hence that there are $[(g-2)(g-3)]2$ holomorphic relations among the elements of Π. The one and only known relation for $g = 4$ preceded Rauch's work by many years and was exhibited by Schottky [13, 14]. Schottky's relation appears as the vanishing of an explicit homogeneous polynomial in the Riemann theta constants. Schottky's derivation of his relation is very complicated and to this day is perhaps still not really understood.

During the last few years there seems to have been a resurgence of interest in the problem. Andreotti and Mayer [1] have proven that there exist polynomials in

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the Riemann theta constants and their derivatives whose vanishing implies all period relations. The author and Rauch in a sequence of papers [2–5, 11, 12] have rederived Schottky's relation for $g = 4$ and have extended it to arbitrary genus. The tool used was essentially the settling of a conjecture by Schottky and Jung in the affirmative. The question has also been considered by Fay in his dissertation [6].

In this paper we restrict our attention to hyperelliptic surfaces and verify directly that for $g = 4$ the period matrices satisfy Schottky's relation. The verification of this fact is also a new proof of the relation and as the reader will see, can be extended in a simple way to hyperelliptic surfaces of arbitrary genus.

I. This section will be devoted to definitions, establishing notation, and summarizing known facts.

Throughout this paper (S,Γ,Δ) will denote a compact Riemann surface of genus g on which there is given a canonical homology basis $\Gamma = \gamma_1,\cdots,\gamma_g$, $\Delta = \delta_1,\cdots,\delta_g$. $\phi_1,\cdots,\phi_g$ will denote the basis of the space of holomorphic abelian differentials on S normalized so that $\int_{\gamma_j}\phi_i = \delta_{i,j}$ and $\Pi = (\pi_{ij})$ where $\pi_{ij} = \int_{\delta_j}\phi_i$. We shall let Φ denote a column vector whose components are $\phi_1,\cdots,\phi_g$. Hence, if we denote the columns of the $g \times g$ identity matrix by $e^{(i)}$, $i = 1,\cdots,g$ and the columns of the $g \times g$ matrix Π by $\Pi^{(i)}$, $i = 1,\cdots,g$ we have $\int_{\gamma_i}\Phi = e^{(i)}$ and $\int_{\delta_i}\Phi = \Pi^{(i)}$.

Definition 1. A period is an integral linear combination of the columns (vectors) $e^{(i)}$ and $\Pi^{(i)}$. We shall denote the period $u_1e^{(i)} + \cdots + u_ge^{(g)} + u_1\Pi^{(1)} + \cdots + u_g\Pi^{(g)}$ by the symbol

$$\left\{\begin{matrix} u \\ u' \end{matrix}\right\} = \left\{\begin{matrix} u_1,\cdots,u_g \\ u_1',\cdots,u_g' \end{matrix}\right\}.$$

A half period is simply half a period. We denote the half period $\frac{1}{2}\left\{\begin{matrix} u \\ u' \end{matrix}\right\}$ by $\begin{pmatrix} u \\ u' \end{pmatrix}$.

If we choose a point p_0 on S and consider the map $\phi: S \to \mathbb{C}^g$ (where $\mathbb{C}^g$ denotes the space of g complex variables) defined by $\phi(p) = \int_{p_0}^{p}\Phi$ we see immediately that ϕ is a holomorphic multivalued map of $S \to \mathbb{C}^g$; however, two images of p can differ only by a period. Hence, if we identify points in $\mathbb{C}^g$ whose difference is a period, the map ϕ becomes single-valued. The map ϕ can be extended in a natural way to S_r the r-fold symmetric product of S with itself. If $p_1\cdots p_r \in S_r$ then $\phi(p_1\cdots p_r)$ is defined to be $\sum_{k=1}^{r}\phi(p_k)$.

Definition 2. A compact Riemann surface of genus g is said to be hyperelliptic provided that there is a meromorphic function on the surface with precisely two poles.

It is well known that a meromorphic function on a compact Riemann surface assumes every value the same number of times and therefore by virtue of definition 2 we see that every compact Riemann surface which is hyperelliptic has a concrete representation as a branched two sheeted cover of the sphere. The genus and the number of branch points are related by the well known Riemann-Hurwitz formula which says that if S_1 and S_2 are Riemann surfaces and f represents S_1 as a branched analytic covering of S_2 then $2g_1 - 2 = n(2g_2 - 2) + V$ where g_i is the genus of S_i, n is the number of sheets and V is the number of branch points of the covering counting multiple order branch points according to their multiplicity. In particular, for hyperelliptic surfaces we have $2g_1 - 2 = 2(-2) + V$ or $V = 2g_1 + 2$. Hence, each hyperelliptic Riemann surface of genus g has a concrete representation as a two sheeted cover of the sphere with $2g + 2$ branch points, and in particular a hyperelliptic surface of genus 4 has a representation as a two sheeted cover of the sphere with ten branch points. We can then by a fractional linear transformation of the sphere arrange that three branch points lie over $0, 1$ and ∞ on the sphere and obtain a concrete representation of a compact Riemann surface of genus 4 as illustrated in Fig. 1.

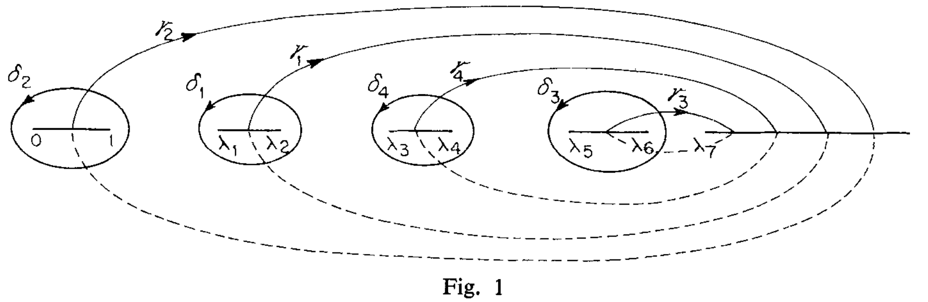

Fig. 1

DEFINITION 3. A theta characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} = \begin{Bmatrix} \varepsilon_1, \cdots, \varepsilon_g \\ \varepsilon'_1, \cdots, \varepsilon'_g \end{Bmatrix}$ is a $2 \times g$ matrix of integers. The characteristic is termed even or odd depending on whether the inner product $\varepsilon \cdot \varepsilon' = \sum_{i=1}^{g} \varepsilon_i \varepsilon'_i$ is even or odd. The characteristic is called a reduced characteristic if the entries in the matrix consist of zeros and ones. The reduced characteristic obtained from a given characteristic by replacing each entry in the matrix by its least non-negative residue modulo two is called the reduced representative of the characteristic and a simple computation shows that the parity of a characteristic and its reduced representative is the same.

DEFINITION 4. Let $Z = (z_1, \cdots, z_g)$ be a vector in $\mathbb{C}^g$ and $\Pi = (\pi_{ij})$ be the $g \times g$ matrix defined at the beginning of this section. (It is well known that Π is symmetric and Im $\Pi \gg 0$). The first order theta function with characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ associated with S, Γ, Δ is defined by the following series:

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(Z,\Pi) = \sum_{n_1 \cdots n_g = -\infty}^{\infty} \exp 2\pi i \left[\tfrac{1}{2} \sum_{i,j=1}^{g} \pi_{ij}(n_i + \varepsilon_{i/2})(n_j + \varepsilon_{j/2}) + \sum_{i=1}^{g} (n_i + \varepsilon_{i/2})(z_i + \varepsilon'_{i/2})\right].$$

If one now takes the first order theta function with characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ associated with S, Γ, Δ and replaces the argument Z by the argument $\int_{p_0}^{p} \Phi$ where p_0 is some fixed point of S one obtains a multivalued holomorphic function on S; however, the zeros of this function are well defined. The latter statement holds since the function $\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(Z,\Pi)$ satisfies the following functional equation:

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}\left(Z + \begin{Bmatrix} u \\ u' \end{Bmatrix}, \Pi\right) = \exp \pi i \left\{ \sum_{i=1}^{g} (\varepsilon_i u'_i - \varepsilon'_i u_i) - 2 \sum_{i=1}^{g} u_i z_i - \sum_{i,j=1}^{g} u_i u_j \pi_{ij} \right\} \theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(Z,\Pi). \tag{1}$$

Therefore, since any two images of p under $\int_{p_0}^{p} \Phi$ differ by a period, a point p on S is unambiguously a zero of the function or not.

Further properties of the theta function that we shall have need of are given by the following formulae:

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}\left(Z + \begin{pmatrix} u \\ u' \end{pmatrix}, \Pi\right) = \exp \pi i \left\{ -\tfrac{1}{4} \sum_{i,j=1}^{g} u_i u_j \pi_{ij} - \tfrac{1}{2} \sum_{i=1}^{g} u_i(\varepsilon'_i + u'_i) - \sum_{i=1}^{g} u_i z_i \right\} \theta\begin{bmatrix} \varepsilon + u \\ \varepsilon' + u' \end{bmatrix}(Z,\Pi). \tag{2}$$

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(Z,\Pi) = (-1)^{\Sigma \varepsilon_i v'_i} \theta\begin{bmatrix} \hat{\varepsilon} \\ \hat{\varepsilon}' \end{bmatrix}(Z,\Pi) \tag{3}$$

where $\hat{\varepsilon} = \varepsilon + 2v$, $\hat{\varepsilon}' = \varepsilon' + 2v'$, and v, v' are vectors in $\mathbb{C}^g$ with integer components.

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(-Z,\Pi)=(-1)^{\Sigma\varepsilon_i\varepsilon'_i}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(Z,\Pi). \tag{4}$$

The theory of theta functions is a vast and beautiful one and can be found in [7]. For a more modern presentation, and one which emphasizes the Riemann surface theory aspect the reader is referred to [8]. The important points for us are the following:

$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\int_{p_0}^{p}\Phi,\Pi)$ is either identically zero on S,Γ,Δ or else has precisely g zeros ($g=$ genus of S) on S,Γ,Δ. Furthermore, in the latter case, if $p_1,\cdots,p_g\in S_g$ denotes the divisor of zeros of $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\int_{p_0}^{p}\Phi,\Pi)$ there is a constant $K\in\mathbb{C}^g$, which depends on p_0 and Γ,Δ, such that $\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix}=\phi(p_1\cdots p_g)+K+\begin{Bmatrix}u\\ u'\end{Bmatrix}$ where ϕ has been defined after definition 1 and $\begin{Bmatrix}u\\ u'\end{Bmatrix}$ is some period as in Definition 1. A point $Z^0=(z_1^0,\cdots,z_g^0)\in\mathbb{C}^g$ is a zero of $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(Z,\Pi)$ if and only if there is an element $\zeta=q_1\cdots q_{g-1}\in S_{g-1}$ such that $Z^0=\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix}+\phi(\zeta)+K+\begin{Bmatrix}u\\ u'\end{Bmatrix}$. In reference to the second point above it would seem as if the property of Z^0 being a zero depended on the base point p_0 of the map ϕ. This of course does not make any sense and in fact the quantity $\phi(\zeta)+K$ for $\zeta\in S_{g-1}$ is independent of the base point p_0 modulo periods [8].

II. For the remainder of this paper we restrict our attention to S,Γ,Δ where S is hyperelliptic surface of genus 4 and Γ,Δ is pictured in Fig 1. We understand Fig. 1 as describing S,Γ,Δ as a two-sheeted covering manifold of the Riemann sphere branched over $0,1,\infty,\lambda_1,\cdots,\lambda_7$ and we shall denote the covering map by Z. Observe, therefore, that the function Z is a meromorphic function on S.

We now recall one of the main theorems of the theory of compact Riemann surfaces, namely Abel's theorem which gives a necessary and sufficient condition for $\zeta\in S_r$ and $\omega\in S_t$ to be the zeros and poles of a meromorphic function on S. (Recall S_r has been defined immediately prior to Definition 2).

ABEL'S THEOREM. A necessary and sufficient condition for $\zeta\in S_r$ and $\omega\in S_t$ (ζ relatively prime to ω) to be the zeros and poles of a meromorphic function on S is that $r=t$ and $\phi(\zeta)-\phi(\omega)=\begin{Bmatrix}u\\ u'\end{Bmatrix}$.

LEMMA 1. *If p, q are two points of S which are inverse images under Z of $0, 1, \infty, \lambda_1, \cdots, \lambda_7$ then* $\phi(p) - \phi(q) = \begin{pmatrix} u \\ u' \end{pmatrix}$.

PROOF. We have already observed that Z is a meromorphic function on S with a double zero and a double pole. Furthermore, $(z-1), (z-\lambda_1), \cdots, (z-\lambda_7)$ have double zeros at $z^{-1}(1)$, $z^{-1}(\lambda_1), \cdots, z^{-1}(\lambda_7)$ and double poles at $z^{-1}(\infty)$. Hence, if p, q are say $z^{-1}(\lambda_i)$ and $z^{-1}(\lambda_j)$ respectively $i \neq j$, $(z-\lambda_i)/(z-\lambda_j)$ is a meromorphic function on S with a double zero at p and a double pole at q. By Abel's theorem $2\phi(p) = \phi(p^2) = \phi(pp) = \phi(qq) = \phi(q^2) = 2\phi(q) + \begin{Bmatrix} u \\ u' \end{Bmatrix}$ and therefore $\phi(p) - \phi(q) = \begin{pmatrix} u \\ u' \end{pmatrix}$ as was to be shown.

If in particular we take $z^{-1}(0)$ to be the base point p_0 of the map ϕ, we find by the previous lemma $\phi(q) = \begin{pmatrix} u \\ u' \end{pmatrix}$, for $q = z^{-1}(1)$, $z^{-1}(\lambda_1), \cdots, z^{-1}(\lambda_7), z^{-1}(\infty)$ and for some u, u'.

As a matter of fact one can characterize hyperelliptic Riemann surfaces by the condition that there exist a base point p_0 for the map ϕ such that $\phi(q) = \begin{pmatrix} u \\ u' \end{pmatrix}$. This follows immediately from the sufficiency part of Abel's theorem.

LEMMA 2. *Choosing $z^{-1}(0)$ to be the base point of the map ϕ we have:*

(1) $\qquad \phi(z^{-1}(0)) = \begin{pmatrix} 0000 \\ 0000 \end{pmatrix}$

(2) $\qquad \phi(z^{-1}(1)) = \begin{pmatrix} 0100 \\ 0000 \end{pmatrix}$

(3) $\qquad \phi(z^{-1}(\lambda_1)) = \begin{pmatrix} 0100 \\ 1100 \end{pmatrix}$

(4) $\qquad \phi(z^{-1}(\lambda_2)) = \begin{pmatrix} 1100 \\ 1100 \end{pmatrix}$

(5) $\qquad \phi(z^{-1}(\lambda_3)) = \begin{pmatrix} 1100 \\ 0101 \end{pmatrix}$

(6) $\qquad \phi(z^{-1}(\lambda_4)) = \begin{pmatrix} 1101 \\ 0101 \end{pmatrix}$

(7) $\qquad \phi(z^{-1}(\lambda_5)) = \begin{pmatrix} 1101 \\ 0110 \end{pmatrix}$

(8) $\phi(z^{-1}(\lambda_6)) = \begin{pmatrix}1111\\0110\end{pmatrix}$

(9) $\phi(z^{-1}(\lambda_7)) = \begin{pmatrix}1111\\0100\end{pmatrix}$

(10) $\phi(z^{-1}(\lambda)) = \begin{pmatrix}0000\\0100\end{pmatrix}$

where we understand equality to mean equality modulo periods in $\mathbb{C}^g$.

PROOF. The proof is by a simple computation which we illustrate by computing $\phi(z^{-1}(\lambda_3))$. Consider a cycle c with initial point at $z^{-1}(0)$ which runs along the top sheet of our surface in Fig. 1 until it comes to $z^{-1}(\lambda_3)$ and then goes back to $z^{-1}(0)$ along the bottom sheet lying under the path already traversed on the top sheet. It is clear that c is homologous to $\gamma_2 + \delta_2 + \delta_1 - \gamma_4$ and hence $\int_c = \begin{Bmatrix}1 & 1 & 0 & 0\\0 & 1 & 0 & -1\end{Bmatrix}$. If we now denote the part of c lying on the top sheet by c^+ and the part of c lying on the bottom sheet by c^- we have $\int_{c^+}\Phi + \int_{c^-}\phi = \begin{Bmatrix}1 & 1 & 0 & 0\\0 & 1 & 0 & -1\end{Bmatrix}$. Utilizing now the well-known fact that dz/ω, zdz/ω, z^2dz/ω, and z^3dz/ω where $\omega = \sqrt{z(z-1)(z-\lambda_1)\cdots(z-\lambda_7)}$ is a basis for the holomorphic differentials on S and the fact that these differentials have the property that they change sign on the two sheets we see that $\int_{c^+}\Phi = \int_{c^-}\Phi$. We immediately have then that $\phi(z^{-1}(\lambda_3)) = \int_{c^+}\Phi = \frac{1}{2}\int_c\Phi = \frac{1}{2}\begin{Bmatrix}1 & 1 & 0 & 0\\0 & 1 & 0 & -1\end{Bmatrix} = \begin{pmatrix}1 & 1 & 0 & 0\\0 & 1 & 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 1 & 0 & 0\\0 & 1 & 0 & 1\end{pmatrix}$ modulo periods. The verification of the other equalities is obtained in precisely the same fashion and this gives us the proof of the lemma.

LEMMA 3. *Choosing $z^{-1}(0)$ to be the base point of the map ϕ we have*

	Function	*Zeros on S, Γ, Δ*
1)	$\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(p),\Pi)$	$z^{-1}(0), z^{-1}(\lambda_2), z^{-1}(\lambda_4), z^{-1}(\lambda_6)$
1′)	$\theta\begin{bmatrix}0100\\0000\end{bmatrix}(\phi(p),\Pi)$	$z^{-1}(\infty), z^{-1}(\lambda_2), z^{-1}(\lambda_4), z^{-1}(\lambda_6)$
2)	$\theta\begin{bmatrix}0100\\0101\end{bmatrix}(\phi(p),\Pi)$	$z^{-1}(0), z^{-1}(\lambda_2), z^{-1}(\lambda_5), z^{-1}(\lambda_7)$
2′)	$\theta\begin{bmatrix}0100\\0001\end{bmatrix}(\phi(p),\Pi)$	$z^{-1}(\infty), z^{-1}(\lambda_2), z^{-1}(\lambda_5), z^{-1}(\lambda_7)$

3) $\theta\begin{bmatrix}0100\\0110\end{bmatrix}(\phi(p),\Pi)$ $z^{-1}(0), z^{-1}(\lambda_2), z^{-1}(\lambda_4), z^{-1}(\lambda_7)$

3′) $\theta\begin{bmatrix}0100\\0010\end{bmatrix}(\phi(p),\Pi)$ $z^{-1}(\infty), z^{-1}(\lambda_2), z^{-1}(\lambda_4), z^{-1}(\lambda_7)$

4) $\theta\begin{bmatrix}0100\\1111\end{bmatrix}(\phi(p),\Pi)$ $z^{-1}(0), z^{-1}(\lambda_3), z^{-1}(\lambda_4), z^{-1}(\lambda_7)$

4′) $\theta\begin{bmatrix}0100\\1011\end{bmatrix}(\phi(p),\Pi)$ $z^{-1}(\infty), z^{-1}(\lambda_3), z^{-1}(\lambda_4), z^{-1}(\lambda_7)$

5) $\theta\begin{bmatrix}0000\\0100\end{bmatrix}(\phi(p),\Pi)$ $z^{-1}(1), z^{-1}(\lambda_2), z^{-1}(\lambda_4), z^{-1}(\lambda_6)$

5′) same as 1′)

6) $\theta\begin{bmatrix}0000\\0101\end{bmatrix}(\phi(p),\Pi)$ $z^{-1}(1), z^{-1}(\lambda_2), z^{-1}(\lambda_5), z^{-1}(\lambda_7)$

6′) same as 2′)

7) $\theta\begin{bmatrix}0000\\0110\end{bmatrix}(\phi(p),\Pi)$ $z^{-1}(1), z^{-1}(\lambda_2), z^{-1}(\lambda_4), z^{-1}(\lambda_7)$

7′) same as 3′)

8) $\theta\begin{bmatrix}0000\\1111\end{bmatrix}(\phi(p),\Pi)$ $z^{-1}(1), z^{-1}(\lambda_3), z^{-1}(\lambda_4), z^{-1}(\lambda_7)$

8′) same as 4′)

PROOF. The proof is by inspection. We verify that the four points indicated are indeed zeros of the given function on S, Γ, Δ and then observe that the given functions are not identically zero on S, Γ, Δ. We verify 1).

Clearly, $\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(z^{-1}(0)),\Pi) = \theta\begin{bmatrix}0100\\0100\end{bmatrix}(0,\Pi) = 0$ by Eq. (4).

$$\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(z^{-1}(\lambda_2),\Pi) = \theta\begin{bmatrix}0100\\0100\end{bmatrix}\left(\begin{pmatrix}1100\\1100\end{pmatrix},\Pi\right) = E\theta\begin{bmatrix}1200\\1200\end{bmatrix}(0,\Pi)$$

by Eq. (2) with $z=0$ where E is an exponential multiplier and this equals by Eq.(3) $\pm\, E\theta\begin{bmatrix}1000\\1000\end{bmatrix}(0,\Pi) = 0$ by Eq. (4). Similarly,

$$\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(z^{-1}(\lambda)_4,\Pi) = \theta\begin{bmatrix}0100\\0100\end{bmatrix}\left(\begin{pmatrix}1101\\0101\end{pmatrix},\Pi\right) = E\theta\begin{bmatrix}1201\\0201\end{bmatrix}(0,\Pi) =$$

$$\pm\, E\theta\begin{bmatrix}1001\\0001\end{bmatrix}(0,\Pi) = 0,$$

and

$$\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(z^{-1}(\lambda_6),\Pi)=\theta\begin{bmatrix}0100\\0100\end{bmatrix}\left(\begin{pmatrix}1111\\0110\end{pmatrix},\Pi\right)=E\theta\begin{bmatrix}1211\\0210\end{bmatrix}(0,\Pi)=$$

$$\pm E\theta\begin{bmatrix}1011\\0010\end{bmatrix}(0,\Pi)=0.$$

The only further item one has to check is that $\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(p),\Pi)$ is not identically zero on S,Γ,Δ when $z^{-1}(0)$ is the base point of the map ϕ. One way of doing this is to check

$$\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(z^{-1}(1)),\Pi)=\theta\begin{bmatrix}0100\\0100\end{bmatrix}\left(\begin{pmatrix}0100\\0000\end{pmatrix},\Pi\right)=E\theta\begin{bmatrix}0200\\0110\end{bmatrix}(0,\Pi)=$$

$$\pm E\theta\begin{bmatrix}0000\\0100\end{bmatrix}(0,\Pi).$$

Now it can be shown quite simply that for S,Γ,Δ there are precisely ten even theta functions which vanish at the origin. The ten are those functions with the following characteristics:

$\begin{bmatrix}1010\\1011\end{bmatrix}$, $\begin{bmatrix}1110\\1011\end{bmatrix}$, $\begin{bmatrix}1110\\0111\end{bmatrix}$, $\begin{bmatrix}0110\\0111\end{bmatrix}$, $\begin{bmatrix}0110\\1110\end{bmatrix}$, $\begin{bmatrix}0111\\1110\end{bmatrix}$, $\begin{bmatrix}0111\\1101\end{bmatrix}$, $\begin{bmatrix}0101\\1101\end{bmatrix}$, $\begin{bmatrix}0101\\1111\end{bmatrix}$, $\begin{bmatrix}1010\\1111\end{bmatrix}$. We therefore conclude that $\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(z^{-1}(1)),\ \Pi)\not\equiv 0$ and hence that the four zeros obtained are the four zeros of the given function on S,Γ,Δ. The verification of the remainder of the assertions is performed in precisely the same fashion.

We now have to make the following observations: Given any two theta functions $\theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(\phi(p),\Pi)$ and $\theta\begin{bmatrix}\delta\\\delta'\end{bmatrix}(\phi(p),\Pi)$ we consider

$$\frac{\theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}^2(\phi(p),\Pi)}{\theta\begin{bmatrix}\delta\\\delta'\end{bmatrix}^2(\phi(p),\Pi)}.$$

This is a meromorphic function on S,Γ,Δ with of course double zeros and poles. In particular, a glance at the data given in Lemma 2 shows that if we take the square of the theta functions listed as $1),\cdots,8)$ and divide them respectively by the squares of the theta functions listed as $1'),\cdots,8')$; the first four quotients have the same zeros and poles and the last four quotients have the same zeros and poles. The first four quotients have a double zero at $z^{-1}(0)$ and a double pole at $z^{-1}(\infty)$ while the last four functions have a double zero at $z^{-1}(1)$ and a double pole at $z^{-1}(\infty)$. The functions z and $z-1$ on the surface however have precisely the same zeros and poles. Hence, the first four quotients must be constant multiples of the function z and the last four must be constant multiples of $z-1$. The constants in question can actually be computed by substituting for p say $z^{-1}(1)$ in the first four quotients and $z^{-1}(0)$ in the last four quotients. We shall not do this yet but rather consider now the quotients without taking squares.

LEMMA 4. *Choosing $z^{-1}(0)$ to be the base point of the map ϕ and appropriate branches of the function* $\sqrt{z}$ and $\sqrt{z-1}$ *we have the following*:

$$1)\quad -\frac{1}{i}\frac{\theta\begin{bmatrix}0000\\0000\end{bmatrix}\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0100\\0000\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z}\text{ and }\frac{\theta\begin{bmatrix}0000\\0000\end{bmatrix}\theta\begin{bmatrix}0000\\1000\end{bmatrix}}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0000\\1100\end{bmatrix}}=\sqrt{\lambda_1}$$

$$2)\quad -\frac{1}{i}\frac{\theta\begin{bmatrix}0000\\0001\end{bmatrix}\theta\begin{bmatrix}0100\\0101\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\0101\end{bmatrix}\theta\begin{bmatrix}0100\\0001\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z}\text{ and }\frac{\theta\begin{bmatrix}0000\\0001\end{bmatrix}\theta\begin{bmatrix}0000\\1001\end{bmatrix}}{\theta\begin{bmatrix}0000\\0101\end{bmatrix}\theta\begin{bmatrix}0000\\1101\end{bmatrix}}=\sqrt{\lambda_1}$$

$$3)\quad -\frac{1}{i}\frac{\theta\begin{bmatrix}0000\\0010\end{bmatrix}\theta\begin{bmatrix}0100\\0110\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\0110\end{bmatrix}\theta\begin{bmatrix}0100\\0010\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z}\text{ and }\frac{\theta\begin{bmatrix}0000\\0010\end{bmatrix}\theta\begin{bmatrix}0000\\1010\end{bmatrix}}{\theta\begin{bmatrix}0000\\0110\end{bmatrix}\theta\begin{bmatrix}0000\\1110\end{bmatrix}}=\sqrt{\lambda_1}$$

$$4)\quad -\frac{1}{i}\frac{\theta\begin{bmatrix}0000\\1011\end{bmatrix}\theta\begin{bmatrix}0100\\1111\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\1111\end{bmatrix}\theta\begin{bmatrix}0100\\1011\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z}\text{ and }\frac{\theta\begin{bmatrix}0000\\1011\end{bmatrix}\theta\begin{bmatrix}0000\\0011\end{bmatrix}}{\theta\begin{bmatrix}0000\\1111\end{bmatrix}\theta\begin{bmatrix}0000\\0111\end{bmatrix}}=\sqrt{\lambda_1}$$

$$5)\quad i\,\frac{\theta\begin{bmatrix}0100\\0000\end{bmatrix}\theta\begin{bmatrix}0000\\0100\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0100\\0000\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z-1}\text{ and }\frac{\theta\begin{bmatrix}0100\\0000\end{bmatrix}\theta\begin{bmatrix}0100\\1000\end{bmatrix}}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0000\\1100\end{bmatrix}}=\sqrt{\lambda_1-1}$$

$$6)\quad i\,\frac{\theta\begin{bmatrix}0100\\0001\end{bmatrix}\theta\begin{bmatrix}0000\\0101\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\0101\end{bmatrix}\theta\begin{bmatrix}0100\\0001\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z-1}\text{ and }\frac{\theta\begin{bmatrix}0100\\0001\end{bmatrix}\begin{bmatrix}0100\\1001\end{bmatrix}}{\theta\begin{bmatrix}0000\\0101\end{bmatrix}\theta\begin{bmatrix}0000\\1101\end{bmatrix}}=\sqrt{\lambda_1-1}$$

7) $$i\,\frac{\theta\begin{bmatrix}0100\\0010\end{bmatrix}\theta\begin{bmatrix}0000\\0110\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\0110\end{bmatrix}\begin{bmatrix}0100\\0010\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z-1}\text{ and }\frac{\begin{bmatrix}0100\\0010\end{bmatrix}\begin{bmatrix}0100\\1010\end{bmatrix}}{\theta\begin{bmatrix}0000\\0110\end{bmatrix}\begin{bmatrix}0000\\1110\end{bmatrix}}=\sqrt{\lambda_1-1}$$

8) $$i\,\frac{\theta\begin{bmatrix}0100\\1011\end{bmatrix}\theta\begin{bmatrix}0000\\1111\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\1111\end{bmatrix}\theta\begin{bmatrix}0100\\1011\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z-1}\text{ and }\frac{\theta\begin{bmatrix}0100\\1011\end{bmatrix}\begin{bmatrix}0100\\0011\end{bmatrix}}{\theta\begin{bmatrix}0000\\1111\end{bmatrix}\begin{bmatrix}0000\\0111\end{bmatrix}}=\sqrt{\lambda_1-1}$$

PROOF. The verification of assertions 1), $\cdots$, 8) all use the same argument which we sketched prior to the statement of the lemma. We shall carry out the details for the first assertion.

Our first task is to observe that the function $\sqrt{z}$ is a two valued function on S,Γ,Δ which has no branching. This is so because z is a function with second order zeros and poles. The nature of the two-valuedness of $\sqrt{z}$ is also clear. $\sqrt{z}$ changes sign on S,Γ,Δ only when z traverses a cycle homologous to $\pm\,\delta_2$.

Our second task is to examine $\dfrac{\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0100\\0000\end{bmatrix}(\phi(p),\Pi)}$ on S,Γ,Δ. The functional Eq. (1) above for the theta functions indicates that the function under consideration is two-valued on S, and moreover that it has precisely the same two valued character as $\sqrt{z}$; i.e., it also changes sign only when p traverses a cycle on S,Γ,Δ homologous to $\pm\,\delta_2$. Hence, we can conclude that there is a constant K such that

$K\dfrac{\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0100\\0000\end{bmatrix}(\phi(p),\Pi)}=\sqrt{z}$. We compute K by letting $p=z^{-1}(1)$ and choosing $\sqrt{1}=1$. We then obtain, using our table in Lemma 2 that $K\dfrac{\begin{bmatrix}0100\\0100\end{bmatrix}\left(\begin{pmatrix}0100\\0000\end{pmatrix},\Pi\right)}{\begin{bmatrix}0100\\0000\end{bmatrix}\left(\begin{pmatrix}0100\\0000\end{pmatrix},\Pi\right)}=1$ or that $K=\dfrac{\theta\begin{bmatrix}0100\\0000\end{bmatrix}\left(\begin{pmatrix}0100\\0000\end{pmatrix},\Pi\right)}{\theta\begin{bmatrix}0100\\0100\end{bmatrix}\left(\begin{pmatrix}0100\\0000\end{pmatrix},\Pi\right)}$. Utilizing now Eq. (2) above with $z=0$ we find $K=\dfrac{\exp\frac{\pi i}{2}[0]\theta\begin{bmatrix}0200\\0000\end{bmatrix}}{\exp\frac{\pi i}{2}[1]\theta\begin{bmatrix}0200\\0100\end{bmatrix}}$ which equals after an application of Eq. (3) $-\dfrac{1}{i}\dfrac{\theta\begin{bmatrix}0000\\0000\end{bmatrix}}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}}$. Hence, we obtain

$\sqrt{z} = -\dfrac{1}{i}\dfrac{\theta\begin{bmatrix}0000\\0000\end{bmatrix}\theta\begin{bmatrix}0100\\0100\end{bmatrix}(\phi(p),\Pi)}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0100\\0000\end{bmatrix}(\phi(p),\Pi)}$. Utilizing now this expression for $\sqrt{z}$ and letting $p = z^{-1}(\lambda_1)$ we find

$$\sqrt{\lambda_1} = -\frac{1}{i}\frac{\theta\begin{bmatrix}0000\\0000\end{bmatrix}\theta\begin{bmatrix}0100\\0100\end{bmatrix}\left(\begin{pmatrix}0100\\1100\end{pmatrix},\Pi\right)}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0100\\0000\end{bmatrix}\left(\begin{pmatrix}0100\\1100\end{pmatrix},\Pi\right)} = -\frac{1}{i}\frac{\theta\begin{bmatrix}0000\\0000\end{bmatrix}\left[\exp\frac{\pi i}{2}[1]\theta\begin{bmatrix}0200\\1200\end{bmatrix}\right.}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\exp\frac{\pi i}{2}[0]\theta\begin{bmatrix}0200\\1100\end{bmatrix}}$$

$= \dfrac{\theta\begin{bmatrix}0000\\0000\end{bmatrix}\theta\begin{bmatrix}0000\\1000\end{bmatrix}}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0000\\1100\end{bmatrix}}$. This concludes the proof of 1) and the rest follow exactly in the same way.

THEOREM. *Let Π be the second half of the period matrix of S, Γ, Δ, S a hyperelliptic surface of genus* 4 *Γ, Δ as pictured in* Fig. 1, then

$$\pm\left[\frac{\theta\begin{bmatrix}0000\\0000\end{bmatrix}\theta\begin{bmatrix}0000\\1000\end{bmatrix}}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0000\\1100\end{bmatrix}}\frac{\theta\begin{bmatrix}0000\\0001\end{bmatrix}\theta\begin{bmatrix}0000\\1001\end{bmatrix}}{\theta\begin{bmatrix}0000\\0101\end{bmatrix}\theta\begin{bmatrix}0000\\1101\end{bmatrix}}\frac{\theta\begin{bmatrix}0000\\0010\end{bmatrix}\theta\begin{bmatrix}0000\\1010\end{bmatrix}}{\theta\begin{bmatrix}0000\\0110\end{bmatrix}\theta\begin{bmatrix}0000\\1110\end{bmatrix}}\frac{\theta\begin{bmatrix}0000\\0011\end{bmatrix}\theta\begin{bmatrix}0000\\1011\end{bmatrix}}{\theta\begin{bmatrix}0000\\0111\end{bmatrix}\theta\begin{bmatrix}0000\\1111\end{bmatrix}}\right]^{\frac{1}{2}}$$

$$\pm\left[\frac{\theta\begin{bmatrix}0100\\0000\end{bmatrix}\theta\begin{bmatrix}0100\\1000\end{bmatrix}}{\theta\begin{bmatrix}0000\\0100\end{bmatrix}\theta\begin{bmatrix}0000\\1100\end{bmatrix}}\frac{\theta\begin{bmatrix}0100\\0001\end{bmatrix}\theta\begin{bmatrix}0100\\1001\end{bmatrix}}{\theta\begin{bmatrix}0000\\0101\end{bmatrix}\theta\begin{bmatrix}0000\\1101\end{bmatrix}}\frac{\theta\begin{bmatrix}0100\\0010\end{bmatrix}\theta\begin{bmatrix}0100\\1010\end{bmatrix}}{\theta\begin{bmatrix}0000\\0110\end{bmatrix}\theta\begin{bmatrix}0000\\1110\end{bmatrix}}\frac{\theta\begin{bmatrix}0100\\0011\end{bmatrix}\theta\begin{bmatrix}0100\\1011\end{bmatrix}}{\theta\begin{bmatrix}0000\\0111\end{bmatrix}\theta\begin{bmatrix}0000\\1111\end{bmatrix}}\right]^{\frac{1}{2}} = \pm 1$$

PROOF. The quantities in the first radical are simply four different expressions for $\sqrt{\lambda_1}$ given in Lemma 4 and the quantities in the second radical are simply the four different expressions for $\sqrt{\lambda_1 - 1}$ given in the same lemma. Hence, the statement of the theorem is simply that

$$\pm\,(\sqrt{\lambda_1}\sqrt{\lambda_1}\sqrt{\lambda_1}\sqrt{\lambda_1})^{\frac{1}{2}} \pm (\sqrt{\lambda_1 - 1}\sqrt{\lambda_1 - 1}\sqrt{\lambda_1 - 1}\sqrt{\lambda_1 - 1})^{\frac{1}{2}} = 1.$$

Our final observation, however, is that the denominators in both radicals are the same. Hence, multiplying through by the common denominator gives precisely the Schottky relation derived in [3]. The procedure used here can be carried over to hyperelliptic surfaces of genus $g > 4$ as well and will always give relations of the form $\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = 0$ with r_i a product of eight theta constants. In [5] a Schottky type relation was obtained for an arbitrary surface of genus 5. The relation there obtained contained four terms. Comparison of the relations shows that the fourth term drops out for hyperelliptic surfaces due to the presence in the product of a vanishing theta constant.

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